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A Sinc-collocation method with boundary treatment for the Stokes equations

Chen Li and Xionghua Wu*,†

Department of Applied Mathematics, Tongji University, 200092 Shanghai, People's Republic of China

SUMMARY

In Stokes equations the velocity u and the pressure p are coupled together by the imcompressibility condition $\operatorname{div}\mathbf{u}=0$ which makes the equations difficult to solve numerically. In this paper, a method named Sinc-collocation method with boundary treatment (SCMBT) is applied to the Stokes equations. The numerical results show that our method is of high accuracy, of good convergence with little computational effort. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The Stokes equations and the Navier–Stokes are of much concern. The Stokes equations describe the stable fluid flow, without nonlinear terms, and the Navier–Stokes equations describe the fluid flow developing with time [1]. By neglecting the term of $\partial u/\partial t$ and the nonlinear convection, the Navier–Stokes equations can be transformed into the Stokes equations which can be regarded as an approximation of the Navier–Stokes equations when the Reynolds number is very low or the velocity is very small.

The Stokes equations discussed in this paper are: ($\Omega=[0,1]\times[0,1]$):

$$\begin{aligned} p_x &= u_{xx}+u_{yy}+f_1 \quad \text{in } \Omega \\ p_y &= v_{xx}+v_{yy}+f_2 \quad \text{in } \Omega \\ u_x+v_y &= 0 \quad \text{in } \Omega \end{aligned} \tag{1}$$

*Correspondence to: Xionghua Wu, Department of Applied Mathematics, Tongji University, 200092 Shanghai, People's Republic of China.
†E-mail: wuxhlu@mail.tongji.edu.cn

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Much work has been done for the Stokes equations, most of which has been completed using the finite element method. Quazzi [2] introduced the vorticity variable, $\omega = \mathrm{curl}\, u$, to study the equations; Bochev and Gunzburger [3] adopted the least squares finite element method to the equations; Belgacem *et al.* [4] studied the mortar spectral element method for the Stokes equations; Jou and Liu [5] gave *a posteriori* finite element error analysis for the Stokes equations.

There are some difficulties in solving the Stokes equations. Firstly, there is evidently no transport or other equation for pressure. The velocity u and the pressure p are coupled together, and this makes the numerical solution difficult. In addition, satisfaction of the continuity equation is not automatic, and the condition should be enforced.

Sinc methods have been studied extensively and found to be a very effective technique, particularly for problems with singular solutions and those on unbounded domains. In addition, Sinc function seem to capture oscillating behaviours in space, hence, are useful to deal with problems characterized by this type of solution [6]. References [7, 8] provide overviews of the methods based on the Sinc function for solving ODE, PDE and integral equations.

In References [9, 10], a method named the Sinc-collocation method with boundary treatment (SCMBT) was introduced, which was applied to two-point initial-boundary value problems and two-dimensional elliptic boundary value problems. It is difficult for the traditional Sinc method to solve two-dimensional elliptic boundary value problems a mixed nonhomogeneous boundary condition [11]. By using SCMBT, no matter what the boundary conditions are, the boundary conditions can be dealt with directly and successfully.

In this paper, SCMBT is applied to the Stokes equations. With our method, we successfully can overcome the difficulty mentioned above. We split the coefficient matrices into block matrices which allow the boundary nodes to be separated from the internal nodes.

To overcome the difficulties, we take the steps as follows. From the first two equations of (1), we can obtain a Poisson equation for p. In addition, the boundary conditions for p_x and p_y can be expressed by functions for u and v from the first two equations of (1). All of these equations are discretized by the SCMBT. Then p_x and p_y can be eliminated in the discretized equations. Thus, the numerical results of u and v can be obtained. Meanwhile, the continuity equation is automatically satisfied by this method.

The numerical results indicate that our method for the Stokes equations is effective. Our method is of high accuracy, simple in principle, easy to program and easy to treat the pressure boundary conditions.

The content of this paper is developed in the four sections, as follows: Section 2 introduces the formulations of SCMBT; Section 3 applies SCMBT to the Stokes equations; Section 4 consists of some numerical examples; and Section 5 is a short conclusion.

2. FORMULATIONS OF SCMBT

2.1. SCMBT formulae in one variable

For a function $w(x)$ on the interval [0, 1], we can get the discrete formulae for $w_{xx} = f_1$, $w_x = f_2$ (for details see References [9, 10]).

Let $\varphi(x)$ be a one-to-one conformal map of interval [0, 1] onto the real line. Here $t = \varphi(x)$ is the double exponential transformation (DE transformation) [12]. Let $\psi = \varphi^{-1}$ denote the

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reverse map, in addition

$$x=\psi(t)=\frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh t\right)+\frac{1}{2}$$

The double exponential formula (DE formula), which is a quadrature formula based on the DE transformation, was first proposed by Takshasi and Mori [12] in 1974. The DE formula has been widely used in the last three decades and is now recognized to be one of the most efficient quadrature formulae [13, 14]. The use of the DE transformation technique in the Sinc method yields a highly efficient numerical method for interpolation, quadrature, approximation of transformation, differential and partial differential equations [14–16].

It is known that the Sinc-collocation method with n collocation points converges at the rate of $\exp(-\kappa\sqrt{n})$ with some $\kappa>0$ under certain condition. From Reference [14], we know that the Sinc-collocation incorporated with the DE transformation converges at the rate of $\exp(-\kappa' n/\log n)$ with some $\kappa'>0$ under rather stringent condition (we have achieved this rate convergence in our numerical examples of Section 4 by our method).

Denote $\mathbf{w}=[w(0), w_x(0), w(x_{-M}),\ldots, w(x_N), w_x(1), w(1)]^{\mathrm{T}}$, then $w_{xx}=f_1$ can be discretized as (for details see References [9, 10])

$$\underline{B}\mathbf{w}=\overline{Q}F_1 \tag{2}$$

where

$$\underline{B}=[B_1, B_2, \widetilde{B}, B_3, B_4],\quad F_1=[f_1(x_{-M-1}),\ldots, f_1(x_{N+1})]^{\mathrm{T}},\quad \overline{Q}=D\left(\frac{1}{\varphi'}\right)$$

$$B_1=\overline{Q}\Phi''_{00}-\widetilde{B}\widetilde{\Phi}_{00},\quad B_2=\overline{Q}\Phi''_{01}-\widetilde{B}\widetilde{\Phi}_{01},\quad B_3=\overline{Q}\Phi''_{11}-\widetilde{B}\widetilde{\Phi}_{11},\quad B_4=\overline{Q}\Phi''_{10}-\widetilde{B}\widetilde{\Phi}_{10}$$

And here set $\varphi_{00}(x)=(2x+1)(1-x)^2$, $\varphi_{10}(x)=x^2(3-2x)$, $\varphi_{01}(x)=x(1-x)^2$, $\varphi_{11}(x)=x^2(x-1)$.

Let $\widetilde{\Phi}_{ij}=[\varphi_{ij}(x_{-M}),\ldots,\varphi_{ij}(x_N)]^{\mathrm{T}}$, $\Phi''_{ij}=[\varphi''_{ij}(x_{-M-1}),\ldots,\varphi''_{ij}(x_{N+1})]^{\mathrm{T}}$, where $i=0,1$, $j=0,1$.

$$\widetilde{B}=\left[\frac{\widetilde{I}^{(2)}}{h_x^2}+D\left(\frac{\varphi''}{(\varphi')^2}\right)\frac{\widetilde{I}^{(1)}}{h_x}+D\left(\frac{1}{\varphi'}\left(\frac{1}{\varphi'}\right)''\right)\widetilde{I}^{(0)}\right]\overline{D(\varphi')} \tag{3}$$

Set $m=M+N+1$. The matrix $\widetilde{B}$ is a $(m+2)\times m$ matrix. The matrices $\widetilde{I}^{(r)}$, $r=0,1,2$ are $(m+2)\times m$ and the diagonal matrices $D(1/\varphi')$, $D(\varphi''/(\varphi')^2)$ and $D(1/\varphi'(1/\varphi')'')$ are $(m+2)\times(m+2)$ matrices. The diagonal matrix $\overline{D(\varphi')}$ is a $m\times m$ matrix. $\widetilde{I}^{(r)}=(\delta^{(r)}_{jk})$, $k=-M,\ldots,N$, and $j=-M-1,-M,\ldots,N,N+1$.

Thus, $\underline{B}$ is a $(m+2)\times(m+4)$ matrix.

Similarly, $w_x=f_2$ can be discretized as (for details see Reference [9])

$$\underline{A}\mathbf{w}=\overline{Q}F_2 \tag{4}$$

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where

$$\underline{A}=[A_1, A_2, \widetilde{A}, A_3, A_4], \quad F_2=[f_2(x_{-M-1}), \ldots, f_2(x_{N+1})]^{\mathrm{T}}$$

$$A_1=\overline{Q}\Phi'_{00}-\widetilde{A}\,\overline{\Phi}_{00}, \quad A_2=\overline{Q}\Phi'_{01}-\widetilde{A}\,\overline{\Phi}_{01}, \quad A_3=\overline{Q}\Phi'_{11}-\widetilde{A}\,\overline{\Phi}_{11}, \quad A_4=\overline{Q}\Phi'_{10}-\widetilde{A}\,\overline{\Phi}_{10}$$

$$\widetilde{A}=\left[-D\left(\frac{1}{\varphi'}\right)\frac{\widetilde{I}^{(1)}}{h_x}+D\left(\frac{1}{\varphi'}\left(\frac{1}{\varphi'}\right)'\right)\widetilde{I}^{(0)}\right]\overline{D((\varphi'))}$$

2.2. SCMBT formulae for general cases in two variables

In this part, we will discuss the discretized formula for $w_y=f_2$, $w_x=f_3$, $w_{yy}=f_4$ and $w_{xx}=f_5$ on $\Omega=[0,1]\times[0,1]$. Here $w(x,y)$ is not bound to be zero on the boundaries. Similarly, they can be discretized as, respectively (for details see References [9, 10]),

$$\underline{Q}_x W\underline{A}_y^{\mathrm{T}}=\overline{Q}_x F_2\overline{Q}_y, \quad \underline{A}_x W\underline{Q}_y=\overline{Q}_x F_3\overline{Q}_y \tag{5}$$

$$\underline{Q}_x W\underline{B}_y^{\mathrm{T}}=\overline{Q}_x F_4\overline{Q}_y, \quad \underline{B}_x W\underline{Q}_y=\overline{Q}_x F_5\overline{Q}_y \tag{6}$$

where $\underline{Q}_x, \underline{Q}_y, \underline{B}_x, \underline{B}_y$ are the same with the matrices obtained by applying our formulae for one variable in x and y directions, respectively.

And $m_x=M_x+N_x+1$, $m_y=M_y+N_y+1$. For simplicity of the expressions, set $M_x=M_y=M$, $N_x=N_y=N$ and $M_x+N_x+1=M_y+N_y+1=M+N+1=m$ without loss of generality.

Thus, we have

$$\underline{A}=\underline{A}_y=\underline{A}_x, \quad \underline{B}=\underline{B}_y=\underline{B}_x, \quad \underline{Q}=\underline{Q}_y=\underline{Q}_x, \quad \overline{Q}=\overline{Q}_y=\overline{Q}_x \tag{7}$$

where $\underline{A}_x, \underline{B}_x, \underline{Q}_x, \overline{Q}_x$ $\underline{A}_y, \underline{B}_y, \underline{Q}_y$ and $\overline{Q}_y$ are the same as those in (5) and (6).

In addition, W in (5) and (6) is defined as

$$W=\begin{bmatrix} w_{00} & w_{y00} & w_{0j} & w_{y01} & w_{01} \\ w_{x00} & w_{xy00} & w_{x0j} & w_{xy01} & w_{x01} \\ w_{i0} & w_{yi0} & \widetilde{W} & w_{yi1} & w_{i1} \\ w_{x10} & w_{xy10} & w_{x1j} & w_{xy11} & w_{x11} \\ w_{10} & w_{y10} & w_{1j} & w_{y11} & w_{11} \end{bmatrix} \tag{8}$$

where $\widetilde{W}=(w(x_l,y_k))_{m\times m}$, $w_{0j}=(w(0,y_k))$, $w_{x0j}=(w_x(0,y_k))$, $w_{x1j}=(w_x(1,y_k))$, $w_{1,j}=(w(1,y_k))$, $w_{i0}=(w(x_l,0))$, $w_{i1}=(w(x_l,1))$, $w_{xi0}=(w_x(x_l,0))$, $w_{xi1}=(w_x(x_l,1))$, $l=-M,\ldots,N$, $k=-M,\ldots,N$, $w_{00}=w(0,0)$, $w_{01}=w(0,1)$, $w_{10}=w(1,0)$, $w_{11}=w(1,1)$, $w_{x00}=w_x(0,0)$, $w_{x01}=w_x(0,1)$, $w_{x10}=w_x(1,0)$, $w_{x11}=w_x(1,1)$, $w_{y00}=w_y(0,0)$, $w_{y01}=w_y(0,1)$, $w_{y10}=w_y(1,0)$, $w_{y11}=w_y(1,1)$, $w_{xy00}=w_{xy}(0,0)$, $w_{xy01}=w_{xy}(0,1)$, $w_{xy10}=w_{xy}(1,0)$ and $w_{xy11}=w_{xy}(1,1)$. Here $\widetilde{W}$ is a $m\times m$ matrix, $w_{0j}, w_{x0j}, w_{x1j}, w_{1,j}$; are row vectors, and $w_{i1}, w_{xi0}, w_{xi1}, w_{i0}$ are column vectors.

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3. APPLYING SCMBT TO THE STOKES EQUATIONS

The Stokes equations discussed here are: ($\Omega = [0, 1] \times [0, 1]$):

$$\begin{aligned} p_x &= u_{xx} + u_{yy} + f_1 \quad \text{in } \Omega \\ p_y &= v_{xx} + v_{yy} + f_2 \quad \text{in } \Omega \\ u_x + v_y &= 0 \quad \text{in } \Omega \end{aligned} \tag{9}$$

In order to express our method clearly, some definitions will first be introduced.

If $D = [d_1, d_2, \ldots, d_k]$ is a matrix and d_i $(i = 1, 2, \ldots, k)$ are $l \times 1$ vectors, denote $\text{Vec}(D)$ as follows:

$$\text{Vec}(D) = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_k \end{bmatrix} \tag{10}$$

$\underline{A}$, $\underline{B}$ and $\underline{Q}$ in (7) are, respectively, split into the following block matrices:

$$\underline{A} = [A_1, A_2, A_3] = \begin{bmatrix} A_{11} & A_{12} & A_{13}^{\mathrm{T}} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33}^{\mathrm{T}} & A_{34} & A_{35} \end{bmatrix} \tag{11}$$

$$\underline{B} = [B_1, B_2, B_3] = \begin{bmatrix} B_{11} & B_{12} & B_{13}^{\mathrm{T}} & B_{14} & B_{15} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} \\ B_{31} & B_{32} & B_{33}^{\mathrm{T}} & B_{34} & B_{35} \end{bmatrix} \tag{12}$$

$$\underline{Q} = [Q_1, Q_2, Q_3] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13}^{\mathrm{T}} & Q_{14} & Q_{15} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} \\ Q_{31} & Q_{32} & Q_{33}^{\mathrm{T}} & Q_{34} & Q_{35} \end{bmatrix} \tag{13}$$

where A_1, A_3, B_1, B_3, Q_1 and Q_3 are columns, A_2, B_2, Q_2 are $(m+2) \times (m+2)$ matrices, A_{13}, A_{33}, A_{21}, A_{22}, A_{24}, A_{25}, B_{13}, B_{33}, B_{21}, B_{22}, B_{24}, B_{25}, Q_{13}, Q_{33}, Q_{21}, Q_{22}, Q_{24} and Q_{25} are $m \times 1$ columns, A_{23}, B_{23}, Q_{23} are $m \times m$ matrices, A_{11}, A_{12}, A_{14}, A_{15}, A_{31}, A_{32}, A_{34}, A_{35}, B_{11}, B_{12}, B_{14}, B_{15}, B_{31}, B_{32}, B_{34}, B_{35}, Q_{11}, Q_{12}, Q_{14}, Q_{15}, Q_{31}, Q_{32}, Q_{34} and Q_{35} are real numbers.

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Like the definition in (8), set

$$U=\begin{bmatrix} u_{00} & u_{y00} & u_{0j} & u_{y01} & u_{01} \\ u_{x00} & u_{xy00} & u_{x0j} & u_{xy01} & u_{x01} \\ u_{i0} & u_{yi0} & \widetilde{U} & u_{yi1} & u_{i1} \\ u_{x10} & u_{xy10} & u_{x1j} & u_{xy11} & u_{x11} \\ u_{10} & u_{y10} & u_{1j} & u_{y11} & u_{11} \end{bmatrix} = \begin{bmatrix} u_1 & u_2^{\mathrm{T}} & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8^{\mathrm{T}} & u_9 \end{bmatrix}, \quad u_5=\begin{bmatrix} \overline{u}_1 & \overline{u}_2^{\mathrm{T}} & \overline{u}_3 \\ \overline{u}_4 & \overline{u}_5 & \overline{u}_6 \\ \overline{u}_7 & \overline{u}_8^{\mathrm{T}} & \overline{u}_9 \end{bmatrix} \tag{14}$$

$$V=\begin{bmatrix} v_{00} & v_{y00} & v_{0j} & v_{y01} & v_{01} \\ v_{x00} & v_{xy00} & v_{x0j} & v_{xy01} & v_{x01} \\ v_{i0} & v_{yi0} & \widetilde{V} & v_{yi1} & v_{i1} \\ v_{x10} & v_{xy10} & v_{x1j} & v_{xy11} & v_{x11} \\ v_{10} & v_{y10} & v_{1j} & v_{y11} & v_{11} \end{bmatrix} = \begin{bmatrix} v_1 & v_2^{\mathrm{T}} & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8^{\mathrm{T}} & v_9 \end{bmatrix}, \quad v_5=\begin{bmatrix} \overline{v}_1 & \overline{v}_2^{\mathrm{T}} & \overline{v}_3 \\ \overline{v}_4 & \overline{v}_5 & \overline{v}_6 \\ \overline{v}_7 & \overline{v}_8^{\mathrm{T}} & \overline{v}_9 \end{bmatrix} \tag{15}$$

$$P=\begin{bmatrix} p_{00} & p_{y00} & p_{0j} & p_{y01} & p_{01} \\ p_{x00} & p_{xy00} & p_{x0j} & p_{xy01} & p_{x01} \\ p_{i0} & p_{yi0} & \widetilde{P} & p_{yi1} & p_{i1} \\ p_{x10} & p_{xy10} & p_{x1j} & p_{xy11} & p_{x11} \\ p_{10} & p_{y10} & p_{1j} & p_{y11} & p_{11} \end{bmatrix}, \quad P_5=\overline{\overline{P}}=\begin{bmatrix} p_{00} & p_{0j} & p_{01} \\ p_{i0} & \widetilde{P} & p_{i1} \\ p_{10} & p_{1j} & p_{11} \end{bmatrix} \tag{16}$$

where $\widetilde{U}=(w(x_l,y_k))_{m\times m}$, $\widetilde{V}=(w(x_l,y_k))_{m\times m}$, $\widetilde{P}=(w(x_l,y_k))_{m\times m}$, $\overline{u}_5=\widetilde{U}, \overline{v}_5=\widetilde{V}$. And $u_1=u_{00}, u_2^{\mathrm{T}}=[u_{y00},u_{0j},u_{y01}], u_3=u_{01}$, $u_7=u_{10}, u_8^{\mathrm{T}}=[u_{y10},u_{1j},u_{y11}], u_9=u_{11}$, $u_4=[u_{x00}, u_{i0}^{\mathrm{T}}, u_{x10}]^{\mathrm{T}}$, $u_6=[u_{x01},u_{i1}^{\mathrm{T}},u_{x11}]^{\mathrm{T}}$, $\overline{u}_1=u_{xy00}$, $\overline{u}_2^{\mathrm{T}}=u_{x0j}$, $\overline{u}_3=u_{xy01}$, $\overline{u}_4=u_{yi0}$, $\overline{u}_6=u_{yi1}$, $\overline{u}_7=u_{xy10}$, $\overline{u}_8^{\mathrm{T}}=u_{x1j}$, $\overline{u}_9=u_{xy11}$, $v_1=v_{00}$, $v_2^{\mathrm{T}}=[v_{y00},v_{0j},u_{y01}]$, $v_3=v_{01}$, $v_7=v_{10}$, $v_8^{\mathrm{T}}=[v_{y10}, v_{1j},u_{y11}]$, $v_9=v_{11}$, $v_4=[v_{x00},v_{i0}^{\mathrm{T}},v_{x10}]^{\mathrm{T}}$, $v_6=[v_{x01},v_{i1}^{\mathrm{T}},v_{x11}]^{\mathrm{T}}$, $\overline{v}_1=v_{xy00}$, $\overline{v}_2^{\mathrm{T}}=v_{x0j}$, $\overline{v}_3=v_{xy01}$, $\overline{v}_4=v_{yi0}$, $\overline{v}_6=v_{yi1}$, $\overline{v}_7=v_{xy10}$, $\overline{v}_8^{\mathrm{T}}=v_{x1j}$ and $\overline{v}_9=v_{xy11}$.

In addition, $u_{00}, u_{y00}, u_{0j}, u_{y01}, u_{01}, u_{x00}, u_{xy00}, u_{x0j}, u_{xy01}, u_{x01}, u_{i0}, u_{yi0}, u_{yi1}, u_{i1}, u_{x10}, u_{xy10}, u_{x1j}, u_{xy11}, u_{x11}, u_{10}, u_{y10}, u_{1j}, u_{y11}, u_{11}, v_{00}, v_{y00}, v_{0j}, v_{y01}, v_{01}, v_{x00}, v_{xy00}, v_{x0j}, v_{xy01}, v_{x01}, v_{i0}, v_{yi0}, v_{yi1}, v_{i1}, v_{x10}, v_{xy10}, v_{x1j}, v_{xy11}, v_{x11}, v_{10}, v_{y10}, v_{1j}, v_{y11}, v_{11}, p_{00}, p_{y00}, p_{0j}, p_{y01}, p_{01}, p_{x00}, p_{xy00}, p_{x0j}, p_{xy01}, p_{x01}, p_{i0}, p_{yi0}, p_{yi1}, p_{i1}, p_{x10}, p_{xy10}, p_{x1j}, p_{xy11}, p_{x11}, p_{10}, p_{y10}, p_{1j}, p_{y11}$ and p_{11} are similar as those definitions in (8).

In this paper, we try to solve the Stokes equations by the SCMBT in the following steps.

Step 1: The aim of this step is to obtain the equations of p_x and p_y.

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By taking the divergence of both sides of the first two equation of (9) and taking into account the condition $\operatorname{div}\mathbf{u}=0$, we obtain

$$\Delta p=\operatorname{div} f$$

i.e.

$$p_{xx}+p_{yy}=f_{1x}+f_{2y} \tag{17}$$

Setting $g=f_{1x}+f_{2y}$, then

$$p_{xx}+p_{yy}=g \tag{18}$$

Step 2: The aim of this step is to get the expression of $\overline{\overline{P}}$ expressed by the elements of U, V defined in (14) and (15).

By (9) and with direct computation, we can have

$$p_{xy}=\frac{f_{1y}+f_{2x}}{2}$$

Thus,

$$\begin{aligned}p_{xy00}&=\frac{f_{1y}(0,0)+f_{2x}(0,0)}{2}, \quad p_{xy01}=\frac{f_{1y}(0,1)+f_{2x}(0,1)}{2}\\ p_{xy10}&=\frac{f_{1y}(1,0)+f_{2x}(1,0)}{2}, \quad p_{xy11}=\frac{f_{1y}(1,1)+f_{2x}(1,1)}{2}\end{aligned} \tag{19}$$

From the first two equations of (9), we get

$$p_x=u_{xx}+u_{yy}+f_1=u_{yy}-v_{xy}+f_1 \tag{20}$$

$$p_y=v_{xx}+v_{yy}+f_2=v_{xx}-u_{xy}+f_2 \tag{21}$$

Discretize (20) by using the formulae (2)–(6) in Section 2 and with the block matrices definitions in (14)–(16), then we have:

$$\begin{aligned}[p_{x00},p_{x0j},p_{x01}]&=[u_{00},u_{y00},u_{0j},u_{y01},u_{01}]\underline{B}^{\mathrm{T}}(\overline{Q}^{\mathrm{T}})^{-1}\\ &\quad-[v_{x00},v_{xy00},v_{x0j},v_{xy01},v_{x01}]\underline{A}^{\mathrm{T}}(\overline{Q}^{\mathrm{T}})^{-1}+F_{10j}\end{aligned} \tag{22}$$

where $F_{10j}=[f_1(0,0),f_1(0,y_{-M}),\ldots,f_1(0,y_N),f_1(0,1)]$.

Similarly, we have:

$$\begin{aligned}[p_{x10},p_{x1j},p_{x11}]&=[u_{10},u_{y10},u_{1j},u_{y11},u_{11}]\underline{B}^{\mathrm{T}}(\overline{Q}^{\mathrm{T}})^{-1}\\ &\quad-[v_{x10},v_{xy10},v_{x1j},v_{xy11},v_{x11}]\underline{A}^{\mathrm{T}}(\overline{Q}^{\mathrm{T}})^{-1}+F_{11j}\end{aligned} \tag{23}$$

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$$\begin{bmatrix} p_{y00} \\ p_{yi0} \\ p_{y10} \end{bmatrix} = (\overline{Q})^{-1}\underline{B}\begin{bmatrix} v_{00} \\ v_{x00} \\ v_{i0} \\ v_{x10} \\ v_{10} \end{bmatrix} - (\overline{Q})^{-1}\underline{A}\begin{bmatrix} u_{y00} \\ u_{xy00} \\ u_{yi0} \\ u_{xy10} \\ u_{y10} \end{bmatrix} + F_{2i0} \tag{24}$$

$$\begin{bmatrix} p_{y01} \\ p_{yi1} \\ p_{y11} \end{bmatrix} = (\overline{Q})^{-1}\underline{B}\begin{bmatrix} v_{01} \\ v_{x01} \\ v_{i1} \\ v_{x11} \\ v_{11} \end{bmatrix} - (\overline{Q})^{-1}\underline{A}\begin{bmatrix} u_{y01} \\ u_{xy01} \\ u_{yi1} \\ u_{xy11} \\ u_{y11} \end{bmatrix} + F_{2i1} \tag{25}$$

where $F_{10j} = [f_1(1,0), f_1(1,y_{-M}), \ldots, f_1(1,y_N), f_1(1,1)]^{\mathrm{T}}$, $F_{2i0} = [f_2(1,0), f_2(x_{-M},0), \ldots, f_2(x_N,0), f_2(1,0)]^{\mathrm{T}}$, and $F_{2i1} = [f_2(1,1), f_2(x_{-M},1), \ldots, f_2(x_N,1), f_2(1,1)]^{\mathrm{T}}$.

Because v_{x00} and v_{x01} can be expressed as $v_{x00} = [1,0,\ldots,0]v_4$, $v_{x01} = [1,0,\ldots,0]v_6$, with the block matrix technique and direct calculation, (22) can be rewritten as the following form:

$$\begin{aligned}[p_{x00}, p_{x0j}, p_{x01}] = {} & \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{1ij} u_i c_{2ij} + \sum_j \sum_{i=2,8} c_{1ij} u_i^{\mathrm{T}} c_{2ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{1ij} \overline{u}_i e_{2ij} \\ & + \sum_j \sum_{i=2,8} e_{1ij} \overline{u}_i^{\mathrm{T}} e_{2ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{3ij} v_i c_{4ij} + \sum_j \sum_{i=2,8} c_{3ij} v_i^{\mathrm{T}} c_{4ij} \\ & + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{3ij} \overline{v}_i e_{4ij} + \sum_j \sum_{i=2,8} e_{3ij} \overline{v}_i^{\mathrm{T}} e_{4ij} + h_1 \end{aligned} \tag{26}$$

where $d_1, c_{1ij}, c_{2ij}, c_{3ij}, c_{4ij}, e_{1ij}, e_{2ij}, e_{3ij}$ and $e_{4ij}, i = 1, \ldots, 4, 6, \ldots, 9$ are known matrices, which can be calculated directly from the known matrices.

Equations (23)–(25) can be rewritten in the similar way:

$$\begin{aligned}[p_{x10}, p_{x1j}, p_{x11}] = {} & \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{5ij} u_i c_{6ij} + \sum_j \sum_{i=2,8} c_{5ij} u_i^{\mathrm{T}} c_{6ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{5ij} \overline{u}_i e_{6ij} \\ & + \sum_j \sum_{i=2,8} e_{5ij} \overline{u}_i^{\mathrm{T}} e_{6ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{7ij} v_i c_{8ij} + \sum_j \sum_{i=2,8} c_{7ij} v_i^{\mathrm{T}} c_{8ij} \\ & + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{7ij} \overline{v}_i e_{8ij} + \sum_j \sum_{i=2,8} e_{7ij} \overline{v}_i^{\mathrm{T}} e_{8ij} + h_2 \end{aligned} \tag{27}$$

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$$\begin{bmatrix} p_{y00} \\ p_{yi0} \\ p_{y10} \end{bmatrix} = \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{9ij} u_i c_{10ij} + \sum_j \sum_{i=2,8} c_{9ij} u_i^{\mathrm{T}} c_{10ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{9ij} \overline{u}_i e_{10ij}$$

$$+ \sum_j \sum_{i=2,8} e_{9ij} \overline{u}_i^{\mathrm{T}} e_{10ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{11ij} v_i c_{12ij} + \sum_j \sum_{i=2,8} c_{11ij} v_i^{\mathrm{T}} c_{12ij}$$

$$+ \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{11ij} \overline{v}_i e_{12ij} + \sum_j \sum_{i=2,8} e_{11ij} \overline{v}_i^{\mathrm{T}} e_{12ij} + h_3 \tag{28}$$

$$\begin{bmatrix} p_{y01} \\ p_{yi1} \\ p_{y11} \end{bmatrix} = \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{13ij} u_i c_{14ij} + \sum_j \sum_{i=2,8} c_{13ij} u_i^{\mathrm{T}} c_{14ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{13ij} \overline{u}_i e_{14ij}$$

$$+ \sum_j \sum_{i=2,8} e_{13ij} \overline{u}_i^{\mathrm{T}} e_{14ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{15ij} v_i c_{16ij} + \sum_j \sum_{i=2,8} c_{15ij} v_i^{\mathrm{T}} c_{16ij}$$

$$+ \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{15ij} \overline{v}_i e_{16ij} + \sum_j \sum_{i=2,8} e_{15ij} \overline{v}_i^{\mathrm{T}} e_{16ij} + h_4 \tag{29}$$

where all of the coefficient matrices are known, which can be calculated directly.

Now, dicretize (18) by (5) and (6), then:

$$\underline{B} P \underline{Q}^{\mathrm{T}} + \underline{Q} P \underline{B}^{\mathrm{T}} = \overline{Q} G \overline{Q}^{\mathrm{T}} \tag{30}$$

where $G = (g(x_i, y_j))_{(m+2)\times(m+2)}$.

Substituting (14)–(16), (19), (26)–(29) and the given p_{00} into the above equation, with direct computation we can get:

$$c_{21} \overline{\overline{P}} c_{22} = \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{17ij} u_i c_{18ij} + \sum_j \sum_{i=2,8} c_{17ij} u_i^{\mathrm{T}} c_{18ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{17ij} \overline{u}_i e_{18ij}$$

$$+ \sum_j \sum_{i=2,8} e_{17ij} \overline{u}_i^{\mathrm{T}} e_{18ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{19ij} v_i c_{20ij} + \sum_j \sum_{i=2,8} c_{19ij} v_i^{\mathrm{T}} c_{20ij}$$

$$+ \sum_j \sum_{i=1, i\neq 2,5,8}^{9} e_{19ij} \overline{v}_i e_{20ij} + \sum_j \sum_{i=2,8} e_{19ij} \overline{v}_i^{\mathrm{T}} e_{20ij} + h_5 \tag{31}$$

where c_{21}, c_{22}, h_5, c_{17ij}, c_{18ij}, c_{19ij}, c_{20ij}, e_{17ij}, e_{18ij}, e_{19ij} and $e_{20ij}, i = 1, \ldots, 4, 6, \ldots, 9$ are known matrices, which can be calculated directly.

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Take Vec(·) on both sides of (31), after calculation we get

$$\mathrm{Vec}(\overline{\overline{P}}) = \sum_{i=1, i\neq 5}^{9} C_i u_i + \sum_{i=1, i\neq 5}^{9} D_i \overline{u}_i + \sum_{i=1, i\neq 5}^{9} E_i v_i + \sum_{i=1, i\neq 5}^{9} F_i \overline{v}_i + \mathrm{Vec}(h_5) \tag{32}$$

where all of the coefficients are known and can be calculated directly.

Remark 1
Without an additional condition for pressure p, there are no unique solutions for p in (9). Because if p_1 is the solution for p in (9), then $p_2 = p_1 + constant$ is also another solution.

Thus, even if u_i, v_i, $\overline{u}_i$ and $\overline{v}_i$ are known, without such additional condition, we still cannot get the solution for $\overline{\overline{P}}$ from (31) uniquely. In addition, (32) is obtained through a pseudoinverse of the matrix computation. So, when u_i, v_i, $\overline{u}_i$ and $\overline{v}_i$ are known, from (32) we can only get a numerical result $\overline{\overline{P}}_1$ for $\overline{\overline{P}}$, which is not unique. Further more, $\overline{\overline{P}}_1 + constant$ is also another solution and the constant can be determined by the additional condition for pressure p, for instance $p(0,0)$ is known.

Step 3: Based on steps 1 and 2, eliminate p.

Dicretize the first two equations of (9) by formulae (5) and (6), then:

$$\underline{A} P \underline{Q}^{\mathrm{T}} = \underline{B} U \underline{Q}^{\mathrm{T}} + \underline{Q} U \underline{B}^{\mathrm{T}} + \overline{Q} F_1 \overline{Q}^{\mathrm{T}} \tag{33}$$

$$\underline{Q} P \underline{A}^{\mathrm{T}} = \underline{B} V \underline{Q}^{\mathrm{T}} + \underline{Q} V \underline{B}^{\mathrm{T}} + \overline{Q} F_2 \overline{Q}^{\mathrm{T}} \tag{34}$$

where $F_1 = (f_1(x_i, y_j))$ and $F_2 = (f_2(x_i, y_j))$.

As what we have done with (30), substitute (14)–(16) and (19), (26)–(29) into (33), and then we have

$$\begin{aligned} c_{23}\overline{\overline{P}}c_{24} = {} & \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{25ij} u_i c_{26ij} + \sum_j \sum_{i=2,8} c_{25ij} u_i^{\mathrm{T}} c_{26ij} + \sum_j \sum_{i=1, i\neq 2,8}^{9} e_{25ij} \overline{u}_i e_{26ij} \\ & + \sum_j \sum_{i=2,8} e_{25ij} \overline{u}_i^{\mathrm{T}} e_{26ij} + \sum_j \sum_{i=1, i\neq 2,5,8}^{9} c_{27ij} v_i c_{28ij} + \sum_j \sum_{i=2,8} c_{27ij} v_i^{\mathrm{T}} c_{28ij} \\ & + \sum_j \sum_{i=1, i\neq 2,8}^{9} e_{27ij} \overline{v}_i e_{28ij} + \sum_j \sum_{i=2,8} e_{27ij} \overline{v}_i^{\mathrm{T}} e_{28ij} + h_6 \end{aligned} \tag{35}$$

where all the coefficients can be calculated directly from above-mentioned matrices.

By taking Vec(·) on both sides of (35), we can get a new equation. Substituting (32) into this new equation, after direct computation, we have

$$H_5 \, \mathrm{Vec}(\widetilde{U}) = \sum_{i=1, i\neq 5}^{9} H_{1i} u_i + \sum_{i=1, i\neq 5}^{9} H_{2i} \overline{u}_i + \sum_{i=1, i\neq 5}^{9} H_{3i} v_i + \sum_{i=1, i\neq 5}^{9} H_{4i} \overline{v}_i + \mathrm{Vec}(h_6) \tag{36}$$

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Define

$$Z_1 = [u_1, u_4^{\mathrm{T}}, u_7, u_2^{\mathrm{T}}, u_8^{\mathrm{T}}, u_3, u_6^{\mathrm{T}}, u_9]^{\mathrm{T}}, \quad Z_2 = [\overline{u}_1, \overline{u}_4^{\mathrm{T}}, \overline{u}_7, \overline{u}_2^{\mathrm{T}}, \overline{u}_8^{\mathrm{T}}, \overline{u}_3, \overline{u}_6^{\mathrm{T}}, \overline{u}_9]^{\mathrm{T}}$$

$$S_1 = [v_1, v_4^{\mathrm{T}}, v_7, v_2^{\mathrm{T}}, v_8^{\mathrm{T}}, v_3, v_6^{\mathrm{T}}, v_9]^{\mathrm{T}}, \quad S_2 = [\overline{v}_1, \overline{v}_4^{\mathrm{T}}, \overline{v}_7, \overline{v}_2^{\mathrm{T}}, \overline{v}_8^{\mathrm{T}}, \overline{v}_3, \overline{v}_6^{\mathrm{T}}, \overline{v}_9]^{\mathrm{T}}$$

Z_1 and S_1 are vectors of $(4m+12) \times 1$, Z_2 and S_2 are vectors of $(4m+4) \times 1$.

Thus (36), i.e. (33), can be expressed as the following form:

$$R_1 \mathrm{Vec}(\widetilde{U}) + R_2 Z_2 + R_3 S_2 = R_4 Z_1 + R_5 S_1 + R_6 \tag{37}$$

In addition, (34) can be rewritten in the similar way:

$$T_1 \mathrm{Vec}(\widetilde{V}) + T_2 Z_2 + T_3 S_2 = T_4 Z_1 + T_5 S_1 + T_6 \tag{38}$$

Step 4: Get the numerical results of u and v.

There are $2(m+4)^2$ elements aggregately in $Z_1, Z_2, S_1, S_2, \mathrm{Vec}(\widetilde{U})$ and $\mathrm{Vec}(\widetilde{V})$. And there are $2(m+2)^2$ equations in (37) and (38) aggregately. If provided the other $2(4m+12)$ equations on the boundaries, we can get the numerical solutions of (37) and (38).

For example, if the boundary conditions are Dirichlet boundary conditions, then Z_1 and S_1 in (37) and (38) are known. Since there are $2(m+2)^2$ unknown elements in $Z_2, S_2, \mathrm{Vec}(\widetilde{U})$ and $\mathrm{Vec}(\widetilde{V})$, we can get the numerical solutions of $Z_2, S_2, \mathrm{Vec}(\widetilde{U})$ and $\mathrm{Vec}(\widetilde{V})$ from (37) and (38) directly.

If the boundary conditions are of the mixed boundary condition type, we can do as follows. From the boundary conditions, we can obtained $2(4m+12)$ equations in related to Z_1, Z_2, S_1 and S_2. Thus with the $2(m+2)^2$ equations in (37) and (38), we can have the numerical solutions $\mathrm{Vec}(\widetilde{U}), \mathrm{Vec}(\widetilde{V}), Z_1, Z_2, S_1$ and S_2.

4. NUMERICAL RESULTS AND ANALYSIS

To verify our method, three examples are treated.

Example 1

In this example, we consider Equation (1) with the Dirichlet boundary conditions.

Its exact solutions are as in Reference [17]:

$$u = (x^4 - 2x^3 + x^2)(4y^3 - 6y^2 + 2y)$$

$$v = -(y^4 - 2y^3 + y^2)(4x^3 - 6x^2 + 2x)$$

$$p = x^5 + y^5$$

Set $E_u = \max|\widetilde{U}_{ij} - u(x_i, y_j)|$, $E_v = \max|\widetilde{V}_{ij} - v(x_i, y_j)|$, E_{u1} is the maximum error of the normal derivatives on the boundaries of u. And E_{v1} is the maximum error of the normal derivatives on the boundaries of v. And the numerical results are shown in Table I. The results of Reference [17] are shown in Table II. Our results are better than those in Reference [17].

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Table I. The errors of U, V of Example 1.

	$M=N=4$	$M=N=5$	$M=N=6$	$M=N=7$	$M=N=8$
E_u	4.2115e−005	2.6542e−005	1.4916e−005	9.7517e−006	6.0942e−006
E_v	4.0414e−005	2.6762e−005	1.4897e−005	9.7517e−006	6.0941e−006

Table II. The errors in Reference [17].

	$N_T=32$	$N_T=64$	$N_T=128$	$N_T=256$	$N_T=512$
E_u	1.16684e−002	5.89109e−003	2.95885e−003	1.48404e−003	7.43309e−004

Table III. The errors of U, V of Example 2 ($m=1$).

	$M=N=4$	$M=N=5$	$M=N=6$	$M=N=7$	$M=N=8$
E_u	3.6063e−004	1.6463e−004	9.5098e−005	6.0142e−005	3.8168e−005
E_v	3.5567e−004	1.8215e−004	9.6404e−005	6.0244e−005	3.7889e−005

Example 2

In this example, we consider Equation (1) with the Dirichlet boundary conditions.

Its exact solutions are:

$$u = \sin(m\pi x)\cos(m\pi y), \quad v = -\cos(m\pi x)\sin(m\pi y), \quad p = \sin(m\pi x)\cos(m\pi y)\,e^{x+y}$$

where m is a integer. In this example, $m=1, 2, 3$ are taken, respectively.

Set E_u and E_v as defined above. With our method when $m=2$, $M=8$, $E_u=5.2003\text{e}{-004}$. When $m=3$, $M=10$, $E_u=3.9292\text{e}{-004}$.

Table III shows the numerical results when $m=1$. In this example, the exact solutions are oscillating functions. The results show that our method can also be applied to such problems.

Example 3

In this example, we consider Equation (1) with the boundary conditions:

$$u=0, v=0 \quad \text{on } \Omega \backslash \{y=1\}$$

$$u+u_y=f_3, v=0 \quad \text{on } \{y=1\}$$

Its exact solutions are

$$u=2x^2(1-x)^2y(1-y)(1-2y), \quad v=-2x(1-x)(1-2x)y^2(1-y)^2$$

$$p=x^2-y^2$$

Thus, f_1, f_2 and f_3 can be calculated. And the numerical results are shown in Table IV.

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Table IV. Only the errors of U, V and their normal derivatives on the boundaries for Example 3.

N	M	E_u	E_v	E_{u1}	E_{v1}
4	4	1.8924e−006	1.2945e−006	1.3791e−004	1.3637e−004
6	6	7.2712e−008	2.2920e−008	4.6084e−005	4.5413e−005
8	8	1.7389e−008	3.8631e−009	1.8668e−005	1.8515e−005

The numerical results of these examples show that in our method the velocity and the pressure can be decoupled easily and successfully, and $u_x + v_y = 0$ can be satisfied automatically. And our method is of high accuracy, of good convergence, simple in principle and convenient to program. The parameters in the DE transformations might depend on the problems and their solutions. For different problems, different parameters might be chosen differently.

5. CONCLUSIONS

The numerical results show that Sinc-collocation method with boundary treatment (SCMBT) for the Stokes equations is effective and by our method the velocity and the pressure can be decoupled easily and successfully. The numerical results show that our method is effective, of high accuracy, of good convergence. Our method is easy to treat the boundary conditions, simple in principle and convenient to implement by programming.

The SCMBT is effective for the Stokes equations. We will try to apply this method to the Navier–Stokes equations though there are some other technique problems which need to be solved. The details for solving the Navier–Stokes equations will be studied in another paper.

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